Euler characteristic. Orientatability

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Theorem

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Sketch of a proof:

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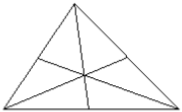

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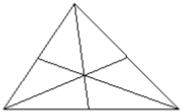

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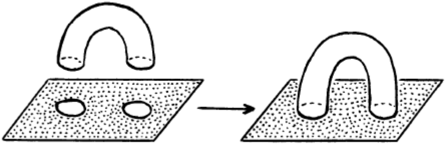
Exercise: compute Euler characteristic of $\mathbb{R}P^2, K^2, T^2, M^2$.

Attaching a Möbius strip

Attaching a handle:



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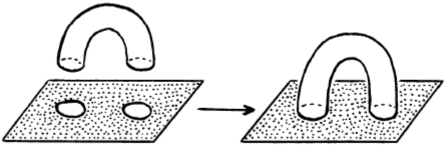
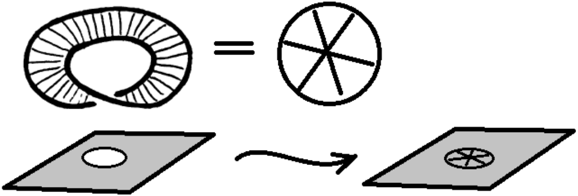


Attaching a Möbius band:



Euler characteristic and attachments

Question: How does $\chi(\Sigma)$ change when attaching a handle? When attaching a Möbius strip?

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If Σ is Σ'' with attached handle, then

$$\chi(\Sigma) = \chi(\Sigma'') + 2$$

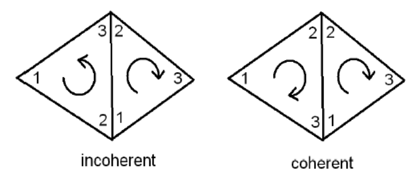
Orientable triangulations

Definition

A **triangulation** of a surface Σ is an embedding of a graph G into Σ such that all faces are triangles.

Definition

A triangulation is **orientable** if all faces can be oriented in a **coherent** way:



Definition

Similarly for any 2-cell decomposition of Σ.

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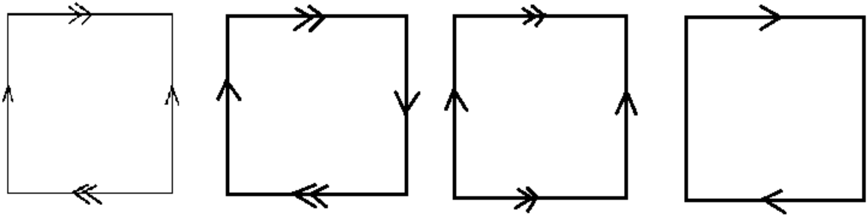
Orientability is invariant under barycentric subdivision.

Orientability is invariant under coarsening.

Which of the following surfaces are orientable?

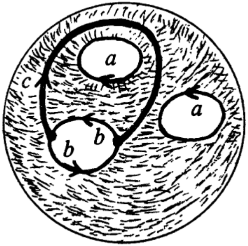


Lemma about attaching

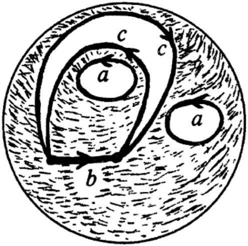
Lemma

Sphere with one handle and one Möbius band is homeomorphic to a sphere with 3 Möbius bands.

Proof:



(a)



(b)